Pluripolar hulls and fine analytic structure

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Abstract

We discuss the relation between pluripolar hulls and fine analytic structure. Our main result is the following. For each non polar subset S of the complex plane $\mathbb{C}$ we prove that there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that the pluripolar hull of E relative to $\mathbb{C}^2$ contains no fine analytic structure and its projection onto the first coordinate plane equals $\mathbb{C}$.

1 Introduction

Denote by Ω an open subset of $\mathbb{C}^n$ and let $E \subset \Omega$ be a pluripolar subset. It might be the case that any plurisubharmonic function $u(z)$ defined in Ω that is equal to $-\infty$ on the set E is necessarily equal to $-\infty$ on a strictly larger set. For instance, if E contains a non polar proper subset of a connected Riemann surface embedded into $\mathbb{C}^n$, then any plurisubharmonic function defined in a neighborhood of the Riemann surface which is equal to $-\infty$ on E is automatically equal to $-\infty$ on the whole Riemann surface. In order to try to understand some aspect of the underlying mechanism of the described "propagation" property of pluripolar sets, the pluripolar hull of graphs $\Gamma_f(D)$ of analytic functions f in a domain $D \subset \mathbb{C}$ has been studied in a number of papers. (See for instance [2], [5], [10] and [14].)

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The pluripolar hull E^*_Ω relative to Ω of a pluripolar set E is defined as follows.

$$E^*_\Omega = \bigcap \{z \in \Omega \ : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions defined in Ω which are equal to $-\infty$ on E. The set E is called *complete pluripolar in* Ω if there exists a plurisubharmonic function on Ω which equals $-\infty$ precisely on E.

As remarked above a necessary condition for a pluripolar set E to satisfy $E^*_\Omega = E$ is that $E \cap A$ is polar in A (or $E \cap A = A$) for all one-dimensional complex analytic varieties $A \subset \Omega$. The fact that this is not a sufficient condition was proved by Levenberg in [8]. By using a refinement of Wermer's example of a polynomially convex compact set with no analytic structure (cf. [13]) Levenberg proved that there exists a compact set $K \subset \mathbb{C}^2$ satisfying $K \neq K^*_{\mathbb{C}^2}$, and the intersection of K with any one dimensional analytic variety A is polar in A. In this example it is not clear what the pluripolar hull $K^*_{\mathbb{C}^2}$ equals.

We will say that a set $S \subset \mathbb{C}^n$ contains fine analytic structure if there exists a non constant map $\varphi : U \to S$ from a fine domain $U \subset \mathbb{C}$ whose coordinate functions are finely holomorphic in U (see Definition 2.3 below). Such a map φ will be called a *fine analytic curve.*

Motivated by recent results of Jöricke and the first author (cf. [5]), the following result was proved in [3].

Theorem 1.1 *Let $\varphi : U \longrightarrow \mathbb{C}^n$ be a finely holomorphic map on a fine domain $U \subset \mathbb{C}$ and let $E \subset \mathbb{C}^n$ be a pluripolar set. Then the following hold*
(1) $\varphi(U)$ is a pluripolar subset of $\mathbb{C}^n$
(2) If $\varphi^{-1}(\varphi(U) \cap E)$ is a non polar subset of $\mathbb{C}$ then $\varphi(U) \subset E^*_{\mathbb{C}^n}$.

In view of this result one may expect to get more information on the pluripolar hull $E^*_{\mathbb{C}^n}$ by examining the intersection of the pluripolar set E with fine analytic curves. Since many curves in $\mathbb{C}^n$ are complete pluripolar (see [4]) one cannot expect that $E^*_{\mathbb{C}^n}$ always contains fine analytic structure. However if we consider the non trivial part $E^*_{\mathbb{C}^n} \setminus E$ the situation is up to now slightly different. In fact, all examples we have seen so far have the property that if $E^*_{\mathbb{C}^n} \setminus E$ is nonempty then for each $w \in E^*_{\mathbb{C}^n} \setminus E$ there exists a finely analytic curve φ contained in $E^*_{\mathbb{C}^n}$ which passes through the point w. (i.e. $\varphi : U \to E^*_{\mathbb{C}^2}$ is a finely analytic

curve and $\varphi(z) = w$ for some $z \in U$). In this paper we prove that no such conclusion holds in general. We have the following main result.

Theorem 1.2 *For each proper non polar subset $S \subset \mathbb{C}$ there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that $E^*_{\mathbb{C}^2}$ contains no fine analytic structure and the projection of $E^*_{\mathbb{C}^2}$ onto the first coordinate plane equals $\mathbb{C}$.*

The set E will be a subset of a complete pluripolar set X which is constructed in the same spirit as Wermer's polynomially convex compact set without analytic structure.
Let us describe more precisely the content of the paper. In Section 2 we briefly recall the construction of Wermer's set and prove that it contains no fine analytic structure. This leads to Theorem 2.4 which slightly generalizes a result in [8]. The main result is proved in Section 3. Subsection 3.1 is devoted to construct the above mentioned set X and in Subsection 3.2 we show that X contains no fine analytic structure. In Subsection 3.3 we define the set E and describe $E^*_{\mathbb{C}^2}$. Finally, in Section 4 we make some remarks and pose two open questions. Readers who are not familiar with basic results on finely holomorphic functions and fine potential theory are referred to [6] and [7].

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2 Wermer's example

In this Section we sketch the details of Wermer's construction given in [13]. Denote by $\mathcal{D}_r$ the open disk with center zero and radius r and by $\mathcal{C}_r$ the open cylinder $\mathcal{D}_r \times \mathbb{C}$. Let $a_1, a_2, ...$ denote the points in the disk $\mathcal{D}_{\frac{1}{2}}$ with rational real and imaginary part. For each j we denote by $B_j(z)$ the algebraic (2-valued) function

$$B_j(z) = (z - a_1)(z - a_2)...(z - a_{j-1})\sqrt{(z - a_j)}.$$

To each n-tuple of positive constants $c_1, c_2, ..., c_n$ we associate the algebraic (2^n-valued) function $g_n(z) = \sum_{j=1}^n c_j B_j(z)$. Let $\sum(c_1, ..., c_n)$, $n = 1, 2, ...$ be the subset of the Riemann surface of $g_n(z)$ which lies in $\overline{\mathcal{C}_{\frac{1}{2}}}$.

Lemma 2.1 *[[13], lemma 1] There exist positive constants $c_1, c_2, ...,$ with $c_1 = \frac{1}{10}$ and $c_{n+1} \leq (\frac{1}{10})c_n$, $n = 1, 2, ...$ and a sequence of polynomials $\{p_n(z, w)\}$ such that:*
(1) $\{p_n = 0\} \cap \{|z| \leq \frac{1}{2}\} = \sum(c_1, ..., c_n)$, $n = 1, 2, ...$
(2) $\{|p_{n+1}| \leq \varepsilon_{n+1}\} \cap \{|z| \leq \frac{1}{2}\} \subset \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}$, $n = 1, 2, ...$
(3) If $|a| \leq \frac{1}{2}$ and $|p_n(a, w)| \leq \varepsilon_n$, then there is a w_n with $p_n(a, w_n) = 0$ and $|w - w_n| \leq \frac{1}{n}$, $n = 1, 2, ...$

With p_n, ε_n, $n = 1, 2, ...$ chosen as in Lemma 2.1, we put

$$Y = \bigcap_{n=1}^{\infty} [\{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}].$$

Clearly, Y is a compact polynomially convex subset of $\mathbb{C}^2$. It was shown by Wermer that Y has no analytic structure i.e. Y contains no non-constant analytic disk. In fact he proves something stronger. The set Y defined above contains no graph of a continuous function defined on a circle in $\mathcal{D}_{\frac{1}{2}}$ which avoids all the branch points $\{a_i\}$. Using this observation the following lemma follows.

Lemma 2.2 *There is no fine analytic curve contained in Y.*

Before we prove Lemma 2.2 we recall the following definition (cf. [7], page 75):

Definition 2.3 *Let U be a finely open set in $\mathbb{C}$. A function $f : U \longrightarrow \mathbb{C}$ is said to be finely holomorphic if every point of U has a compact (in the usual topology) fine neighbourhood $K \subset U$ such that the restriction $f|_K$ belongs to $R(K)$.*

Here $R(K)$ denotes the uniform closure of the algebra of all restrictions to K of rational functions on $\mathbb{C}$ with poles off K.

Proof of Lemma 2.2. Let $\varphi : U \to Y$, $z \mapsto (\varphi_1(z), \varphi_2(z))$ be a fine analytic curve contained in Y. If $\varphi_1(z)$ is constant on U then $\varphi_2(z)$ must also be a constant since non constant finely holomorphic functions are finely open maps and by the construction of the set Y the fibre $Y \cap (\{z\} \times \mathbb{C})$ is a Cantor set or a finite set for any point $z \in \overline{\mathcal{D}}_{1/2}$. Assume therefore that $\varphi_1(z)$ is non-constant. In particular, there is a point $z_0 \in U$ where the fine derivative of $\varphi_1(z)$ does not vanish. Hence $\varphi_1(z)$ is one-to-one on some finely open neighborhood $V \subset U$ of the point z_0. By considering the map $z \mapsto (\varphi_1 \circ \varphi_1^{-1}(z), \varphi_2 \circ \varphi_1^{-1}(z))$, defined on the finely open set $\varphi_1(V)$ we may assume that φ is of the form $z \mapsto (z, g(z))$ where

$g(z) = \varphi_2 \circ \varphi_1^{-1}(z)$ is finely holomorphic in the finely open set $V' = \varphi_1(V) \subset \mathcal{D}_{1/2}$. By Definition 2.3 there exists a compact subset $K \subset V'$ with non-empty fine interior such that $g(z)$ is a continuous function on K (with respect to the Euclidean topology). Shrinking K if necessary we may assume that $K \cap \{a_1, a_2, ...\} = \emptyset$. Let p be a point in the fine interior of K. It is well known that there exists a sequence of circles $\{C(p, r_j)\}$ contained in K with centers p and radii $r_j \to 0$ as $j \to \infty$. Clearly, the circle $C(p, r_j)$ avoids the branch points $\{a_1, a_2, ...\}$ and its image under the continuous map $z \mapsto (z, g(z))$ is contained in Y. By the above observation this is not possible. Hence Y contains no fine analytic structure. $\square$

Denote by d_n the degree of the one variable polynomial $w \mapsto p_n(z, w)$ where $p_n(z, w)$ is the polynomial given in Lemma 2.1. Assume that the set Y is constructed using the parameters ϵ_n satisfying the following condition

$$\lim_{n\to\infty} (\epsilon_n)^{1/d_n} = 0. \tag{1}$$

It is shown in [9] that with this choice the set $Y \cap \mathcal{C}_{1/2}$ is complete pluripolar in $\mathcal{C}_{1/2}$. Using this result and Lemma 2.2 we are able to generalize a result in [8].

Theorem 2.4 *Fix $\delta \in (0, 1/2)$ and let $Y_\delta = \bigcap_{n=1}^{\infty}[\{|p_n| \le \varepsilon_n\} \cap \{|z| \le \delta\}]$ be constructed using the parameters ε_n satisfying (1). Then*
(a) $\varphi^{-1}(\varphi(U) \cap Y_\delta)$ is a polar subset of U for all fine analytic curves $\varphi : U \to \mathbb{C}^2$.
(b) $Y_\delta \neq (Y_\delta)^*_{\mathbb{C}^2}$.

Proof of Theorem 2.4. In order to prove (a) we argue by contradiction. Assume therefore that $\varphi : U \to \mathbb{C}^2$ is a fine analytic curve and $\varphi^{-1}(\varphi(U) \cap Y_\delta)$ is a non polar subset of U. Then there is a fine domain $U_{k_0} \subseteq U$ such that $\varphi(U_{k_0}) \subset \mathcal{C}_{1/2}$ and $\varphi^{-1}(\varphi(U_{k_0}) \cap Y_\delta)$ is non polar. Indeed, the set $\varphi^{-1}(\varphi(U) \cap \mathcal{C}_{1/2})$ is a finely open subset of U and hence has at most countably many finely connected components $\{U_k\}_{k=1}^{\infty}$. Moreover, $\varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0}$ is non polar for some natural number k_0, since otherwise $\bigcup_{k=1}^{\infty}\{\varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_k\} = \varphi^{-1}(\varphi(U) \cap Y_\delta)$ would be polar contrary to our assumption. Since $Y \cap \mathcal{C}_{1/2}$ is complete pluripolar in $\mathcal{C}_{1/2}$ there exists a plurisubharmonic function u defined in $\mathcal{C}_{1/2}$ which is equal to $-\infty$ exactly on $Y \cap \mathcal{C}_{1/2}$. The function $u \circ \varphi$ is either finely subharmonic on U_{k_0} or identically equal to $-\infty$ (cf. [3], Lemma 3.1). Since u equals $-\infty$ on the non polar subset $\varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0}$, it must be identically equal to $-\infty$ on U_{k_0}. Therefore $\varphi(U_{k_0}) \subset \{u = -\infty\} = Y \cap \mathcal{C}_{1/2}$ contradicting Lemma 2.2 and (a) follows.

The proof of assertion (b) follows immediately from the proof of Proposition 3.1 in [8]. Indeed, if u is a plurisubharmonic function defined in $\mathbb{C}^2$ which equals $-\infty$ on Y_δ then the function $z \mapsto \max\{u(z,w) : (z,w) \in Y\}$ is subharmonic in $\mathcal{D}_{1/2}$ and since it equals $-\infty$ on $\mathcal{D}_\delta$ it equals $-\infty$ on $\mathcal{D}_{1/2}$. Consequently $Y \cap \mathcal{C}_{1/2} \subset (Y_\delta)^*_{\mathbb{C}^2}$ and hence $Y_\delta \neq (Y_\delta)^*_{\mathbb{C}^2}$. $\square$

Remark. It follows from the argument used in the proof of assertion (b) in Theorem 2.4 that $Y \cap \mathcal{C}_{1/2} \subset (Y_\delta)^*_{\mathcal{C}_{1/2}}$. Since the first set is complete pluripolar in $\mathcal{C}_{1/2}$ it follows that $(Y_\delta)^*_{\mathcal{C}_{1/2}} = Y \cap \mathcal{C}_{1/2}$. Consequently, $(Y_\delta)^*_{\mathcal{C}_{1/2}}$ contains no fine analytic structure. It would be nice to determine what the set $(Y_\delta)^*_{\mathbb{C}^2}$ equals and to figure out whether this set contains fine analytic structure. We are unable to do this. But by modifying Wermer's construction, we will in the next Section construct a complete pluripolar Wermer-like set $X \subset \mathbb{C}^2$ with the property that $(X \cap (S \times \mathbb{C}))^*_{\mathbb{C}^2}$ contains no fine analytic structure for all non polar subset $S \subset \mathbb{C}$.

3 Proof of Theorem 1.2

3.1 Construction of the set X

In this Subsection we construct the set X. Denote by $\{a_k\}_{k=1}^{\infty}$ the points in the complex plane both of whose coordinates are rational numbers. Without loss of generality we may assume that $a_k \in \mathcal{D}_k$. For any sequence of points $\{a_l\}_{l=1}^{j}$ we denote by $B_j(z)$ the algebraic function

$$B_j(z) = (z - a_1) \dots (z - a_{j-1}) \sqrt{(z - a_j)}.$$

Denote by γ_j a simple smooth curve with endpoints a_j and ∞. For each j $B_j(z)$ has two single-valued analytic branches on $\mathbb{C} \setminus \gamma_j$. Following the notation in [13] we choose one of the branches $B_j(z)$ arbitrarily and denote it by $\beta_j(z)$. Then $|\beta_j(z)| = |B_j(z)|$ is continuous on $\mathbb{C}$.

For each $n+1$-tuple of positive constants $(c_1, c_2, \dots, c_{n+1})$ we denote by $g_n(z)$ the algebraic function defined recursively in the following way. Put $g_1(z) = c_1 B_1(z)$ and $g_2(z) = c_1 B_1(z) + c_2 B_2(z)$ and if $g_n(z)$ has been chosen we will choose $g_{n+1}(z)$ as described below. Put $Z_1(z) = 1$ and for $n = 2, 3, \dots$ define the function $Z_n(z)$ as follows. Denote by $z_1, z_2, \dots, z_l$ all the zeros

of all possible different differences $h_j(z)-h_i(z)$ $(i \neq j)$ of branches $h_i(z), h_j(z)$ of the function $g_n(z)$. Suppose z_k is a zero of $h_j(z)-h_i(z)$ of order m_k and put $Z_n(z) = \Pi_{i=1}^{l}(z-z_i)^{m_i}$. Note that the zeros of $Z_n(z)$ are also zeros of the function $Z_{n+1}(z)$ of the same or greater multiplicity. Define $g_{n+1}(z) = g_n(z) + c_{n+1}Z_n(z)B_{n+1}(z)$.

By $\Sigma(c_1,\dots,c_n)$ we mean the Riemann surface of $g_n(z)$ which lies in $\mathbb{C}^2$. In other words, $\Sigma(c_1,\dots,c_n) = \{(z,w) : z \in \mathbb{C}, w = w_j, j = 1,2,\dots,2^n\}$, where $w_j,\ j = 1,2,\dots,2^n$ are the values of $g_n(z)$ at z.

We will choose positive constants c_n, ϵ_n and polynomials $p_n(z,w)$ recursively so that

$$\{p_n(z,w)=0\} \cap \mathcal{C}_{n+1} = \Sigma(c_1,c_2,\dots,c_n) \cap \mathcal{C}_{n+1} \text{ and} \tag{2}$$

$$\{|p_{n+1}(z,w)| \le \epsilon_{n+1}\} \cap \mathcal{C}_{n+1} \subset \{|p_n(z,w)| < \epsilon_n\} \cap \mathcal{C}_{n+1} \tag{3}$$

hold for $n = 1,2,\dots$. The set X will be of the form

$$X = \bigcup_{n=1}^{\infty}\Big(\bigcap_{j=n}^{\infty}\{|p_j(z,w)| \le \epsilon_j\} \cap \mathcal{C}_{n+1}\Big). \tag{4}$$

Put $c_1 = 1$ and let $p_1(z,w) = w^2-(z-a_1)$. It is clear that $\Sigma(c_1)\cap\mathcal{C}_2 = \{p_1(z,w)=0\}\cap\mathcal{C}_2$. Choose $\epsilon_1 > 0$ so that if $z_0 \in \mathcal{D}_2$ and $|p_1(z_0,w)| \le \epsilon_1$ then there exists $(z_0,w_1) \in \Sigma(c_1)\cap\mathcal{C}_2$ with $|w-w_1| \le 1$. Let $\mathcal{B}_2 = \mathcal{D}_2 \times \mathcal{D}_{\rho_1}$ be a bidisk where ρ_1 is chosen so that

$$\{|p_1(z,w)| \le \epsilon_1\} \cap \mathcal{C}_2 = \{|p_1(z,w)| \le \epsilon_1\} \cap \mathcal{B}_2.$$

Assume that c_n, ϵ_n and $p_n(z,w)$ have been chosen so that (2) and (3) hold. We will now choose c_{n+1} and $p_{n+1}(z,w)$. We denote by $w_j(z),\ j = 1,2,\dots,2^n$ the roots of $p_n(z,\cdot) = 0$ and to each positive constant c we assign a polynomial $p_c(z,w)$ by putting

$$p_c(z,w) = \Pi_{j=1}^{2^n}\Big((w-w_j(z))^2 - c^2(Z_n(z)B_{n+1}(z))^2\Big). \tag{5}$$

Then $p_c(z,\cdot) = 0$ has the roots $w_j(z) \pm cZ_n(z)B_{n+1}(z)$, $j = 1,2,\dots,2^n$ and so

$$\{p_c(z,w) = 0\} = \Sigma(c_1,c_2,\dots,c_n,c).$$

Note that from (5)

$$p_c = p_n^2 + c^2 q_1 + \dots + (c^2)^{2^n} q_{2^n},$$

where the q_j are polynomials in z and w, not depending on c. Choose $c > 0$ so that

$$\Sigma(c_1, c_2, \ldots, c_n, c) \cap \mathcal{C}_{n+1} \subset \{|p_n(z,w)| < \epsilon_n/2\} \cap \mathcal{C}_{n+1} \text{ and} \tag{6}$$

$$c \cdot |Z_n(z)B_{n+1}(z)| \le (1/10)c_n|Z_{n-1}(z)B_n(z)| \text{ holds for all } z \in \mathcal{D}_{n+1}. \tag{7}$$

Decreasing c if necessary we may assume that if $h_i(z)$ and $h_j(z)$ are any different branches of the function $g_n(z)$ the estimate

$$|h_j(z) - h_i(z)| \ge 2c|Z_n(z)B_{n+1}(z)| \tag{8}$$

holds in $\mathcal{D}_{n+1}$ with equality exactly at the zeros of $Z_n(z)$ which are contained in $\mathcal{D}_{n+1}$ and at the points $a_1, \ldots a_n$. This estimate will be needed later when we prove that X contains no fine analytic structure. Choose $c_{n+1} = c$.

Let $\mathcal{B}_{n+2} = \mathcal{D}_{n+2} \times \mathcal{D}_{\rho_{n+2}}$ be a bidisk where ρ_{n+2} is chosen so that $\{|p_n(z,w)| \le \epsilon_n\} \cap \mathcal{C}_{n+2} = \{|p_n(z,w)| \le \epsilon_n\} \cap \mathcal{B}_{n+2}$ and $\rho_{n+2} > \rho_{n+1} + 1$. Let $\delta > 0$ be a constant such that $|\delta \cdot p_c(z,w)| < 1$ in $\mathcal{B}_{n+2}$ and choose $p_{n+1}(z,w) = \delta \cdot p_c(z,w)$.

We now turn to the choice of ϵ_{n+1}. Since the part of the zero set of $p_{n+1}(z,w)$ which is contained in $\mathcal{B}_{n+1}$ is a subset of $\{|p_n(z,w)| < \epsilon_n/2\} \cap \mathcal{B}_{n+1}$ it is possible to find a natural number m_{n+1} so that

$$\frac{1}{m_{n+1}} \log|p_{n+1}(z,w)| \ge -\frac{1}{2^n} \text{ for all } (z,w) \in \mathcal{B}_{n+1} \smallsetminus \{|p_n(z,w)| \le \epsilon_n\}. \tag{9}$$

Choose $\epsilon_{n+1} < \epsilon_n$ so that

$$\frac{1}{m_{n+1}} \log|p_{n+1}(z,w)| \le -1 \text{ for all } (z,w) \in \{|p_{n+1}(z,w)| \le \epsilon_{n+1}\} \cap \mathcal{C}_{n+2}. \tag{10}$$

By decreasing ϵ_{n+1} we may assume that (3) and the following assumption hold.

$$\begin{aligned} &\text{If } (z_0,w) \in \mathcal{C}_{n+2} \text{ and } |p_{n+1}(z_0,w)| \le \epsilon_{n+1}, \text{ then there exists} \\ &(z_0,w_n) \in \mathcal{C}_{n+2} \text{ such that } |p_{n+1}(z_0,w_n)| = 0 \text{ and } |w - w_n| \le 1/n. \end{aligned} \tag{11}$$

This ends the recursion.

Lemma 3.1 *The set X defined by (4) is complete pluripolar in $\mathbb{C}^2$.*

Proof. Define for $n \ge 2$ the plurisubharmonic function

$$u_n(z,w) = \max\Big\{\frac{1}{m_n}\log|p_n(z,w)|, -1\Big\}$$

and put $u(z,w) = \sum_{n\geq 2} u_n(z,w)$. Then $u(z,w)$ is plurisubharmonic in $\mathbb{C}^2$. Indeed, since the bidisks $\mathcal{B}_n$ exhaust $\mathbb{C}^2$ and $|p_n(z,w)| < 1$ in $\mathcal{B}_{n+1}$ the series $\sum_{n\geq 2} u_n(z,w)$ will be decreasing on each fixed bidisk $\mathcal{B}_N$ after a finite number of terms and hence plurisubharmonic there. Since plurisubharmonicity is a local property $u(z,w)$ is plurisubharmonic in $\mathbb{C}^2$. If $(z_0,w_0) \in X$, then for some natural number N, $(z_0,w_0) \in \bigcap_{j=N}^{\infty}\{|p_j(z,w)| \leq \epsilon_j\} \cap \mathcal{C}_{N+1}$. Condition (10) above implies that $u(z_0,w_0) = Const + \sum_{n>N} u_n(z_0,w_0) = -\infty$. Finally if $(z_0,w_0) \notin X$ then there exists a natural number N such that $(z_0,w_0) \in \mathcal{B}_N$ and $(z_0,w_0) \notin \{|p_n(z,w)| \leq \epsilon_n\} \cap \mathcal{B}_N$ for all $n \geq N$. By (9)

$$u(z,w) = Const + \sum_{n>N} \max\Big\{\frac{1}{m_n}\log|p_n(z,w)|, -1\Big\} \geq Const + \sum_{n>N} -\frac{1}{2^n} > -\infty.$$

The Lemma follows. $\square$

3.2 X contains no fine analytic structure

In this Section we show that X contains no fine analytic structure. Suppose that $z \mapsto (\varphi_1(z), \varphi_2(z))$ is a fine analytic curve whose image is contained in X. If $\varphi_1(z)$ is constant then $\varphi_2(z)$ must be constant since $X \cap (\{z_0\} \times \mathbb{C})$ is a Cantor set or a finite set for any point $z_0 \in \mathbb{C}$. On the other hand, if $\varphi_1(z)$ is non-constant, then using the arguments given in the proof of Lemma 2.2 we may assume that the fine analytic curve contained in X is given by $z \mapsto (z, m(z))$ where $m(z)$ is a finely holomorphic function defined in U where $U \subset \mathcal{D}_n$ for some natural number n. Fix a point $z' \in U \setminus \{a_1, \ldots, a_n\}$. By the definition of finely holomorphic functions we can find a compact (in the usual topology) fine neighborhood $K \subset U$ of z' where $m(z)$ is continuous. Shrinking K if necessary we may assume that $(K \setminus \{z'\}) \cap (\{a_j\}_{j=1}^{\infty} \cup \{Z_{k-1}(z) = 0\}_{k=2}^{\infty}) = \emptyset$. Since the complement of K is thin at z', one can find a sequence of circles $\{C(z', r_i)\} \subset K$ with $r_i \to 0$ as $i \to \infty$. Choose one of the circles $C(z', r_j)$ so that none of the points $a_1, \ldots, a_n$ are contained in $\{|z - z'| \leq r_j\}$. Let a_k be the first point in the sequence $\{a_j\}_{j=n+1}^{\infty}$ which is contained in $\{|z - z'| \leq r_j\}$. Note that $a_k \in \{|z - z'| < r_j\}$, $m(z)$ is continuous on $C(z', r_j)$ and the function $Z_{k-1}(z)\beta_k(z) \neq 0$ when $z \in C(z', r_j)$. The fact that the image of $C(z', r_j)$ under the map $z \mapsto (z, m(z))$ is a subset of X will lead us to a contradiction and hence X contains no fine analytic structure. In order to prove this fix a point $z_1 \in C(z', r_j)$ and denote by $\Re$ the 2^k branches of the algebraic function $g_k(z)$ defined on $C(z', r_j) \setminus \{z_1\}$.

Lemma 3.2 *If $h_i(z)$ and $h_j(z)$ are any different functions from $\Re$ then*

$$|h_i(z) - h_j(z)| > (3/2)c_k|Z_{k-1}(z)\beta_k(z)| \tag{12}$$

holds for all $z \in C(z', r_j) \setminus \{z_1\}$.

Proof. This is follows directly from (8) since $C(z', r_j) \subset \mathcal{D}_n$ and $C(z', r_j)$ does not intersect any of the branch points $a_1, \ldots, a_k$ or the zeros of $Z_{k-1}(z)$. □

From now on the proof that X contains no fine analytic structure follows the arguments given in [13].

Lemma 3.3 *Fix z_0 in $C(z', r_j) \setminus \{z_1\}$. There exists a function $h_i(z) \in \Re$, where $h_i(z)$ depends on z_0 such that*

$$|m(z_0) - h_i(z_0)| < (1/4)c_k|Z_{k-1}(z_0)\beta_k(z_0)| \tag{13}$$

Proof. By (11) there exists $N \geq k$ and w_N such that (z_0, w_N) lies on $\Sigma(c_1, \ldots, c_N)$ and $m(z_0) = w_N + R(z_0)$ where $|R(z_0)| \leq (1/10)c_k|Z_{k-1}(z_0)\beta_k(z_0)|$. Thus

$$\begin{aligned} m(z_0) &= \pm c_1\beta_1(z_0) + \sum_{\nu=2}^{N} \pm c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0) = \\ &\stackrel{\text{def}}{=} h_i(z_0) + \sum_{\nu=k+1}^{N} c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0). \end{aligned}$$

Since $C(z', r_j) \subset \mathcal{D}_{n+1}$ and the constants c_ν are chosen so that (7) holds,

$$\begin{aligned} |m(z_0) - h_i(z_0)| &\leq \sum_{\nu=k+1}^{N} c_\nu |Z_{\nu-1}(z_0)\beta_\nu(z_0)| + |R(z_0)| \leq \\ &\leq c_k|Z_{k-1}(z_0)\beta_k(z_0)|(\frac{1}{10} + \frac{1}{10^2} + \ldots) + |R(z_0)| = \\ &= \frac{1}{9}c_k|Z_{k-1}(z_0)\beta_k(z_0)| + \frac{1}{10}c_k|Z_{k-1}(z_0)\beta_k(z_0)| < \\ &< (1/4)c_k|Z_{k-1}(z_0)\beta_k(z_0)|. \end{aligned}$$

Hence (13) holds and the Lemma is proved. □

Lemma 3.4 *Fix $z_0 \in C(z', r_j) \smallsetminus \{z_1\}$ and let $h_i(z) \in \mathfrak{R}$ satisfy (13). Then for all z in $C(z', r_j) \smallsetminus \{z_1\}$*

$$|m(z) - h_i(z)| < (1/3)c_k|Z_{k-1}(z)\beta_k(z)|. \tag{14}$$

Proof. The set $\mathcal{O} = \{z \in C(z', r_j) \smallsetminus \{z_1\} : (14) \text{ holds at } z\}$ is open in $C(z_0, r_j) \smallsetminus \{z_1\}$ and contains z_0. If $\mathcal{O} \neq C(z', r_j) \smallsetminus \{z_1\}$ then there is a boundary point p of $\mathcal{O}$ on $C(z', r_j) \smallsetminus \{z_1\}$ for which

$$|m(p) - h_i(p)| = (1/3)c_k|Z_{k-1}(p)\beta_k(p)| \tag{15}$$

holds. By Lemma 3.3 there is some $h_j(z)$ in $\mathfrak{R}$ such that

$$|m(p) - h_j(p)| < (1/4)c_k|Z_{k-1}(p)\beta_k(p)|. \tag{16}$$

Thus $|h_i(p) - h_j(p)| \leq (7/12)c_k|Z_{k-1}(p)\beta_k(p)|$. Also $h_i(z) \neq h_j(z)$, in view of (15) and (16). This contradicts Lemma 3.2. Thus $\mathcal{O} = C(z', r_j) \smallsetminus \{z_1\}$ and Lemma 3.4 follows. □

For each continuous function $v(z)$ defined on $C(z', r_j) \smallsetminus \{z_1\}$ which has a jump at z_1 we write $L^+(v)$ and $L^-(v)$ for the two limits of $v(z)$ as $z \to z_1$ along $C(z', r_j)$. Then, by (14),

$$|L^+(m) - L^+(h_i)| \leq (1/3)c_k|Z_{k-1}(z_1)\beta_k(z_1)|$$

and

$$|L^-(m) - L^-(h_i)| \leq (1/3)c_k|Z_{k-1}(z_1)\beta_k(z_1)|,$$

so

$$|(L^+(m) - L^+(h_i)) - (L^-(m) - L^-(h_i))| \leq (2/3)c_k|Z_{k-1}(z_1)\beta_k(z_1)|.$$

Since $m(z)$ is continuous on $C(z', r_j)$ the jump of $h_i(z)$ at z_1 is in modulus less than or equal to $(2/3)c_k|Z_{k-1}(z_1)\beta_k(z_1)| \neq 0$. But $h_i(z)$ is in $\mathfrak{R}$, so its jump at z_1 has modulus at least $2c_k|Z_{k-1}(z_1)\beta_k(z_1)|$. This is a contradiction.

3.3 The sets E and $E^*_{\mathbb{C}^2}$

Denote by E the pluripolar set $E = (S \times \mathbb{C}) \cap X$ where S is a non polar subset of $\mathbb{C}$. Since X is complete pluripolar in $\mathbb{C}^2$ it follows that $E^*_{\mathbb{C}^2} \subset X$. To prove that $X \subset E^*_{\mathbb{C}^2}$ we argue as follows. First we claim that the set X is pseudoconcave. Indeed, by the construction of the set X,

$$\mathbb{C}^2 \smallsetminus X = \cup_{n=1}^{\infty} \{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}. \tag{17}$$

By the choice of the polynomials $p_n(z,w)$ it follows that

$$\{|p_n(z,w)| > \epsilon_n\} \cap \mathcal{C}_{n+1} \subset \{|p_{n+1}(z,w)| > \epsilon_{n+1}\} \cap \mathcal{C}_{n+2}.$$

Moreover, for each natural number n the set $\{|p_n(z,w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}$ is a domain of holomorphy. Hence $\mathbb{C}^2 \setminus X$ is a countable union of increasing domains of holomorphy. By the Behnke-Stein Theorem $\mathbb{C}^2 \setminus X$ is pseudoconvex and the claim follows.

Denote by $u(z,w)$ a globally defined plurisubharmonic function which equals $-\infty$ on E. It is shown in [12] that the function $z \mapsto \max\{u(z,w) : (z,w) \in X\}$ is subharmonic in $\mathbb{C}$. Since the projection S of E onto the first coordinate plane is non polar the function $z \mapsto \max\{u(z,w) : (z,w) \in X\}$ will be identically equal to $-\infty$ on $\mathbb{C}$ hence $u(z,w) = -\infty$ on the whole of X and consequently $E^*_{\mathbb{C}^2} = X$. This ends the proof of Theorem 1.2.

4 Final remarks and open problems

It follows immediately from Theorem 1.1 and the fact that X contains no fine analytic structure that if $\varphi : U \to \mathbb{C}^2$ is a fine analytic curve, then the set $\varphi^{-1}(\varphi(U) \cap X)$ is polar in $\mathbb{C}$.

Despite the result of Theorem 1.2 it should be mentioned here that in the situation where one considers the pluripolar hull of the graph of a finely holomorphic function defined in a fine domain D, the following problem still remains open.

Problem 1. Let $z \in \Gamma_f(D)^*_{\mathbb{C}^2}$. Does this imply that there is a fine analytic curve contained in $\Gamma_f(D)^*_{\mathbb{C}^2}$ which passes through the point z?

It is proved in [2] that the pluripolar hull relative to $\mathbb{C}^n$ of a connected pluripolar F_σ subset is a connected set. It is a fairly easy exercise to show that the set $X = E^*_{\mathbb{C}^2}$ in Theorem 1.2 is path connected, but in general the pluripolar hull of a connected (F_σ) pluripolar set is *not* path connected. Indeed, denote by $f(z)$ an entire function of order 1/3. $f(1/z)$ has an essential singularity at 0 and in [14] Wiegerinck proved that the graph $\Gamma_{f(1/z)}$ of $f(1/z)$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$. Consequently, if we put $E = \Gamma_{f(1/z)} \cup (\{0\} \times \mathbb{C})$ then E is complete pluripolar in $\mathbb{C}^2$ and hence $E^*_{\mathbb{C}^2} = E$. Moreover E is a connected F_σ subset of $\mathbb{C}^2$. By the famous Denjoy-Carleman-Ahlfors theorem (see e.g. [1]), entire functions of

order 1/3 do not have finite asymptotic values; i.e., there are no curves γ ending at infinity such that $f(z)$ approaches a finite value as $z \to \infty$ along γ. Hence it is not possible to find a path in $E^*_{\mathbb{C}^2}$ connecting a point on $\Gamma_{f(1/z)}$ with a point in the set $\{0\} \times \mathbb{C}$. In view of this remark it would be interesting to know the answer to the following question.

Problem 2. Is $\Gamma_f(D)^*_{\mathbb{C}^2}$ path connected ?

Finally, we mention here again the following problem from [3].
Problem 3. Let K be a compact set in $\mathbb{C}^n$ and suppose that $\varphi^{-1}(K \cap \varphi(U))$ is a polar subset of U (or empty) for any fine analytic curve $\varphi : U \longrightarrow \mathbb{C}^n$. Must K be a pluripolar subset of $\mathbb{C}^n$?

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